Inference About a Common Mean of Independent Normals

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Examples

Example 1: Meier (1953) (reanalyzed in Jordan and Krishnamoorthy, 1996) considered four experiments about the percentage of albumin in plasma protein in human subjects

Percentage of albumin in plasma protein

Experiment	n_i	Mean	Variance
A	12	62.3	12.986
B	15	60.3	7.840
C	7	59.5	33.433
D	16	61.5	18.513

Examples

Example 2: Eberhardt, Reeve, and Spiegelman (1989) estimated mean Selenium in nonfat milk powder by combining the results of four methods

Selenium in nonfat milk powder

Methods	n_i	Mean	Variance
Atomic absorption spectrometry	8	105.0	85.711
Neutron activation:			
1). Instrumental	12	109.75	20.748
2). Radiochemical	14	109.5	2.729
Isotope dilution mass spectrometry	8	113.25	33.640

Zacks, 1966 (JASA), 1970 (AMS)

The best of my papers were motivated by consulting problems.... In 1963, I was approached by a soil engineer. He wanted to estimate the common mean of two populations and he didn't know anything about the variances. But, a priori from his theory he said that the means should be same, and here are the two samples from two different soils. So I thought about this problem a little bit and I started to investigate. I realized that there is room for innovation (Kempthorne et al., 1991).

Common Mean Problem

- Let us consider k independent normal populations where the ith population follows a normal distribution with mean $\mu \in I\!R$ and variance $\sigma_i^2 > 0$, $i = 1, \ldots, k$.

- Let $\bar{Y}_i$ denote the sample mean in the ith population, S_i^2 the sample variance, and n_i the sample size, $i = 1, \ldots, k$.

- Then, we have

$$\bar{Y}_i \sim N\left(\mu\ ,\ \frac{\sigma_i^2}{n_i}\right) \quad \text{and} \quad \frac{(n_i - 1)\ S_i^2}{\sigma_i^2} \sim \chi^2_{n_i - 1},\ i = 1, \ldots, k,$$

  and the statistics are all mutually independent.
  Note that $(\bar{Y}_i, S_i^2, i = 1, \ldots, k)$ is minimal sufficient for $(\mu, \sigma_1^2, \ldots, \sigma_k^2)$ even though it is not complete.

Estimates of μ

- If the population variances $\sigma_1^2, \ldots, \sigma_k^2$ are completely known, the maximum likelihood estimator of μ is given by

$$\hat{\mu} = \frac{\sum_{i=1}^{k} \frac{n_i}{\sigma_i^2} \bar{Y}_i}{\sum_{j=1}^{k} \frac{n_j}{\sigma_j^2}}.$$

- The above estimator is also the minimum variance unbiased estimator under normality as well as the best linear unbiased estimator without normality for estimating μ.

- The variance of $\hat{\mu}$ is given by $\quad \text{Var}\,(\hat{\mu}) = \frac{1}{\sum_{i=1}^{k} \frac{n_i}{\sigma_i^2}}.$

Estimates of μ

- Graybill-Deal (1959) estimator of μ is given as

$$\hat{\mu}_{GD} = \frac{\sum_{i=1}^{k} \frac{n_i}{S_i^2} \bar{Y}_i}{\sum_{j=1}^{k} \frac{n_j}{S_j^2}}.$$

  Clearly, $\hat{\mu}_{GD}$ is an unbiased estimator of the common mean μ.

- For calculating the variance of $\hat{\mu}_{GD}$, it holds

$$\begin{aligned} \mathsf{Var}\left(\hat{\mu}_{GD}\right) &= \mathsf{E}\left[\mathsf{Var}\left(\hat{\mu}_{GD}|S_1,\ldots,S_k\right)\right] + \mathsf{Var}\left[\mathsf{E}\left(\hat{\mu}_{GD}|S_1,\ldots,S_k\right)\right] \\ &= \mathsf{E}\left[\left(\sum_{i=1}^{k} \frac{n_i\ \sigma_i^2}{S_i^4}\right) \Big/ \left(\sum_{i=1}^{k} \frac{n_i}{S_i^2}\right)^2\right]. \end{aligned}$$

Estimates of μ

- Exact variance expression: Khatri and Shah (CS, 1974)
- Exact distribution: Nair (AS, 1980)

Meier (1953) derived a first order approximation of the variance of $\hat{\mu}_{GD}$ as

$$\text{Var}\left(\hat{\mu}_{GD}\right) = \frac{1}{\sum_{i=1}^{k} \frac{n_i}{\sigma_i^2}} \left[1 + 2\sum_{i=1}^{k} \frac{1}{n_i - 1}\, c_i\, (1 - c_i) + O\left(\sum_{i=1}^{k} \frac{1}{(n_i - 1)^2}\right)\right]$$

with

$$c_i = \frac{n_i \,/\, \sigma_i^2}{\sum_{j=1}^{k} n_j \,/\, \sigma_j^2},\ \ i = 1, \ldots, k.$$

Estimates of μ: Is pooling of data sets necessary?

- Is $\hat{\mu}_{GD}$ a uniformly better unbiased estimator of μ than is each $\bar{Y}_i$, $i = 1, \ldots, k$? Is Var$(\hat{\mu}_{GD}) \leq \sigma_i^2/n_i$, $i = 1, \ldots, k$ for all $\sigma_1^2, \ldots, \sigma_k^2$?

- Graybill and Deal (1959) showed for $k = 2$ that $\hat{\mu}_{GD}$ is a uniformly better unbiased estimator of μ than is $\bar{Y}_1$ or $\bar{Y}_2$ if and only if n_1 and n_2 are each greater than 10.

- Norwood and Hinkelmann (1977): $\hat{\mu}_{GD}$ is a uniformly better estimator of μ than each $\bar{Y}_i$ if and only if each sample size n_i, $i = 1, \ldots, k > 2$, is greater than 10 or $n_i = 10$ for some i and n_j greater than 18 for all $j \neq i$.

Variance Estimates

Sinha (1985) derived an unbiased estimator of the variance of $\hat{\mu}_{GD}$ that is a convergent series. A first order approximation of this estimator is

$$\widehat{\text{Var}}_{(1)}\left(\hat{\mu}_{GD}\right) = \frac{1}{\sum_{i=1}^{k} \frac{n_i}{S_i^2}} \left[1 + \sum_{i=1}^{k} \frac{4}{n_i + 1} \left(\frac{n_i \;/\; S_i^2}{\sum_{j=1}^{k} n_j \;/\; S_j^2} - \frac{n_i^2 \;/\; S_i^4}{\left(\sum_{j=1}^{k} n_j \;/\; S_j^2\right)^2} \right) \right].$$

This estimator is comparable to Meier's (1953) approximate estimator:

$$\widehat{\text{Var}}_{(2)}\left(\hat{\mu}_{GD}\right) = \frac{1}{\sum_{i=1}^{k} \frac{n_i}{S_i^2}} \left[1 + \sum_{i=1}^{k} \frac{4}{n_i - 1} \left(\frac{n_i \;/\; S_i^2}{\sum_{j=1}^{k} n_j \;/\; S_j^2} - \frac{n_i^2 \;/\; S_i^4}{\left(\sum_{j=1}^{k} n_j \;/\; S_j^2\right)^2} \right) \right].$$

Variance Estimates

Two further estimators of the variance of $\hat{\mu}_{GD}$ which can be easily adapted for later purposes.

- The "classical" meta-analysis variance estimator

$$\widehat{\mathsf{Var}}_{(3)}\left(\hat{\mu}_{GD}\right) = \frac{1}{\sum_{i=1}^{k} \frac{n_i}{S_i^2}}.$$

- Hartung (1999): approximate variance estimator

$$\widehat{\mathsf{Var}}_{(4)}\left(\hat{\mu}_{GD}\right) = \frac{1}{k-1} \sum_{i=1}^{k} \frac{n_i \;/\; S_i^2}{\sum_{j=1}^{k} n_j \;/\; S_j^2} \left(\bar{Y}_i - \hat{\mu}_{GD}\right)^2.$$

Approximate Confidence Intervals

- A simple large sample $100(1-\alpha)\%$ confidence interval, which is widely used in meta-analysis, is given by

$$\mathsf{CI}_{(1)}(\mu):\ \hat{\mu}_{GD} \mp \sqrt{\frac{1}{\sum_{i=1}^{k} \frac{n_i}{S_i^2}}}\, z_{\alpha/2}.$$

- This interval, however, mostly proves to be too narrow and the actual confidence coefficient of $\mathsf{CI}_{(1)}(\mu)$ can be dramatically less than the nominal one, see Li, Shi, and Roth (1994) and Böckenhoff and Hartung (1998).

Approximate Confidence Intervals

- Based on concavity corrections for the estimates of $1/\sigma_i^2$, $i = 1, \ldots, k$, and following the lines of $\text{CI}_{(1)}(\mu)$, Böckenhoff and Hartung (1998) worked out improved confidence intervals for μ.

- A larger coverage probability can also be achieved by using the upper $(\alpha/2)$ critical values of a t-distribution with ν degrees of freedom, say $t_{\nu;\alpha/2}$, instead of $z_{\alpha/2}$. Follmann and Proschan (1999) suggested the choice of $\nu = k - 1$ degrees of freedom.

Approximate Confidence Intervals

Meier (1953) showed that the distribution of $\widehat{\mathsf{Var}}_{(2)}(\hat{\mu}_{GD})$ can be approximated by a scaled χ^2-distribution with estimated degrees of freedom $\hat{\nu}$, where

$$\frac{1}{\hat{\nu}} = \sum_{i=1}^{k} \frac{1}{n_i - 1} \left(\frac{n_i \ / \ S_i^2}{\sum_{j=1}^{k} n_j \ / \ S_j^2} \right)^2 .$$

Using the same approximate distribution for $\widehat{\mathsf{Var}}_{(1)}(\hat{\mu}_{GD})$, two approximate $100(1-\alpha)\%$ confidence intervals on μ are given as

$$\mathsf{CI}_{(2)}(\mu): \ \hat{\mu}_{GD} \mp \sqrt{\widehat{\mathsf{Var}}_{(1)}(\hat{\mu}_{GD})} \ t_{\hat{\nu};\alpha/2}$$

and

$$\mathsf{CI}_{(3)}(\mu): \ \hat{\mu}_{GD} \mp \sqrt{\widehat{\mathsf{Var}}_{(2)}(\hat{\mu}_{GD})} \ t_{\hat{\nu};\alpha/2}.$$

Approximate Confidence Intervals

Finally, an approximate $100(1-\alpha)\%$ confidence interval for μ, that does not require the estimation of degrees of freedom, can be constructed using the variance estimator $\widehat{\text{Var}}_{(4)}(\hat{\mu}_{GD})$.
Since, suitably scaled, $\widehat{\text{Var}}_{(4)}(\hat{\mu}_{GD})$ can be well approximated by a χ^2-distribution with $k-1$ degrees of freedom, an approximate $100(1-\alpha)\%$ confidence interval for μ is given as

$$\text{CI}_{(4)}(\mu): \ \hat{\mu}_{GD} \mp \sqrt{\widehat{\text{Var}}_{(4)}(\hat{\mu}_{GD})}\ t_{k-1;\alpha/2}.$$

Exact Confidence Intervals

- Why should we use approximate confidence intervals? A lot of exact intervals exist in the common mean problem.

- Since
$$t_i = \frac{\sqrt{n_i}\ \left(\bar{Y}_i - \mu\right)}{S_i} \sim t_{n_i - 1}$$
or, equivalently,
$$F_i = \frac{n_i\ \left(\bar{Y}_i - \mu\right)^2}{S_i^2} \sim F_{1, n_i - 1}$$
are test statistics for testing hypotheses about μ based on the ith sample, suitable linear combinations of these test statistics or other functions thereof can be used as a pivotal quantity to construct exact confidence intervals for μ.

Exact Confidence Intervals

- Cohen and Sackrowitz (1984) considered $M_t = \max_{1 \le i \le k}\{|t_i|\}$.

- Determine $c_{\alpha/2}$, which satisfies the following equation

$$1 - \alpha = \mathsf{P}\left(M_t \le c_{\alpha/2}\right) = \prod_{i=1}^{k} \mathsf{P}\left(|t_i| \le c_{\alpha/2}\right).$$

- An exact $100(1-\alpha)\%$ confidence interval for μ is then given by

$$\begin{aligned} \mathsf{CI}_{(5)}(\mu): \quad & \left[\max_{1\le i\le k}\left\{\bar{Y}_i - \frac{c_{\alpha/2}\, S_i}{\sqrt{n_i}}\right\},\ \min_{1\le i\le k}\left\{\bar{Y}_i + \frac{c_{\alpha/2}\, S_i}{\sqrt{n_i}}\right\}\right] \\ = \quad & \bigcap_{i=1}^{k}\left[\bar{Y}_i - \frac{c_{\alpha/2}\, S_i}{\sqrt{n_i}},\ \bar{Y}_i + \frac{c_{\alpha/2}\, S_i}{\sqrt{n_i}}\right]. \end{aligned}$$

Exact Confidence Intervals

- An alternative approach is to use the confidence interval

$$\mathsf{CI}_{(6)}(\mu):\quad \left[\max_{1\le i\le k}\left\{\bar{Y}_i - \frac{c^{(i)}_{\alpha/2}\, S_i}{\sqrt{n_i}}\right\}\ ,\ \min_{1\le i\le k}\left\{\bar{Y}_i + \frac{c^{(i)}_{\alpha/2}\, S_i}{\sqrt{n_i}}\right\}\right]$$

$$= \bigcap_{i=1}^{k}\left[\bar{Y}_i - \frac{c^{(i)}_{\alpha/2}\, S_i}{\sqrt{n_i}}\ ,\ \bar{Y}_i + \frac{c^{(i)}_{\alpha/2}\, S_i}{\sqrt{n_i}}\right]\ ,$$

where $c^{(i)}_{\alpha/2}$ satisfies the equation

$$\mathsf{P}\left(|t_i| \le c^{(i)}_{\alpha/2}\right) = (1-\alpha)^{1/k}.$$

Exact Confidence Intervals

Fairweather (1972): weighted linear combination of the t_i's, namely

$$W_t = \sum_{i=1}^{k} u_i \; t_i, \quad u_i = \frac{\left[\mathsf{Var}(t_i)\right]^{-1}}{\sum_{j=1}^{k}\left[\mathsf{Var}(t_i)\right]^{-1}}, i = 1, \ldots, k.$$

Let $b_{\alpha/2}$ denote the upper critical value of the distribution of W_t satisfying the equation $1 - \alpha = \mathsf{P}\left(|W_t| \leq b_{\alpha/2}\right)$, then the exact $100(1-\alpha)\%$ confidence interval for μ is given by

$$CI_{(7)}(\mu) : \; \frac{\sum_{i=1}^{k}\sqrt{n_i}\; u_i \; \bar{Y}_i \; / \; S_i}{\sum_{i=1}^{k}\sqrt{n_i}\; u_i \; / \; S_i} \mp \frac{b_{\alpha/2}}{\sum_{i=1}^{k}\sqrt{n_i}\; u_i \; / \; S_i}.$$

Exact Confidence Intervals

Jordan and Krishnamoorthy (1996): weighted linear combination of the F-test statistics F_i, namely

$$W_f = \sum_{i=1}^{k} w_i \ F_i \ , \ w_i = \frac{\left[\mathsf{Var}(F_i)\right]^{-1}}{\sum_{j=1}^{k} \left[\mathsf{Var}(F_j)\right]^{-1}}, \ i = 1, \ldots, k.$$

Note that $\mathsf{Var}(F_i) = 2 \ m_i^2 \ (m_i - 1)/[(m_i - 2)^2 \ (m_i - 4)]$ with $m_i = n_i - 1$, $i = 1, \ldots, k$.

Determine a_α satisfying

$$1 - \alpha = \mathsf{P}\left(W_f \le a_\alpha\right).$$

Exact Confidence Intervals

- An exact $100(1-\alpha)\%$ confidence interval for μ is given as

$$CI_{(8)}(\mu) : \sum_{i=1}^{k} p_i \, \bar{Y}_i \mp \Delta,$$

where

$$p_i = \frac{w_i \, n_i \, / \, S_i^2}{\sum_{j=1}^{k} w_j \, n_j \, / \, S_j^2}, \; i = 1, \ldots, k,$$

and

$$\Delta^2 = \frac{a_\alpha}{\sum_{i=1}^{k} w_i \, n_i \, / \, S_i^2} - \left\{ \sum_{i=1}^{k} p_i \, \bar{Y}_i^2 - \left(\sum_{i=1}^{k} p_i \, \bar{Y}_i \right)^2 \right\}.$$

Exact Confidence Intervals

- Yu, Sun, and Sinha (1999) derived exact $100(1-\alpha)\%$ confidence intervals for μ using P-values of the F-test statistics F_i. Recall that F_i is a F_{1,n_i-1}-distributed random variable, then the ith P-value P_i is defined as
$$P_i = \int_{F_i}^{\infty} h_i(x)\mathrm{d}x$$
where $h_i(x)$ denotes the probability density function of the F-distribution with 1 and $(n_i - 1)$ degrees of freedom. Note that $P_1, \ldots, P_k$ are independently uniformly distributed random variables.

- Combination methods: Inverse Normal Method and Fisher's Method

- Note that by using Tippett's minimum P value method, one obtains the interval $CI_{(6)}(\mu)$, see Yu, Sun, and Sinha (1999).

Exact Confidence Intervals

Using the inverse normal method, hypotheses about μ will be rejected if

$$\frac{\sum_{i=1}^{k} \Phi^{-1}(P_i)}{\sqrt{k}} < -z_\alpha,$$

where Φ^{-1} denotes the inverse of the cdf of the standard normal distribution. Consequently, an exact $100(1-\alpha)\%$ confidence interval for μ is given by inverting the acceptance region, that is,

$$\text{CI}_{(9)}(\mu) : \left\{ \mu : \frac{\sum_{i=1}^{k} \Phi^{-1}(P_i)}{\sqrt{k}} > -z_\alpha \right\}.$$

Exact Confidence Intervals

Using Fisher's inverse χ^2-method, hypotheses about μ will be rejected if

$$-2\sum_{i=1}^{k}\ln(P_i) > \chi^2_{2k;\alpha},$$

where $\chi^2_{2k;\alpha}$ denotes the upper α critical value of a χ^2-distribution with $2\ k$ degrees of freedom. Again, by inverting the acceptance region, we obtain an exact $100(1-\alpha)\%$ confidence interval for μ as

$$\mathsf{CI}_{(10)}(\mu):\ \left\{\mu:\ -2\sum_{i=1}^{k}\ln(P_i) < \chi^2_{2k;\alpha}\right\}.$$

Exact Confidence Intervals

- We have six(!) exact confidence intervals. Which one should we use?

- All the intervals except Fairweather's interval do not necessarily produce a genuine interval.

- Yu, Sun, and Sinha (1999) derived sufficient conditions for the inverse χ^2-method and the inverse normal method to produce genuine intervals.

Exact Confidence Intervals

- In a simulation study for $k = 2$ populations, Yu, Sun, and Sinha (1999) showed that the interval with the inverse χ^2-method outperforms the other P-value based exact confidence intervals for μ in terms of expected length.

- Compared to the other exact intervals, they recommended the use of Fairweather's interval, when the two population variances are close and small, followed by the interval with inverse χ^2-method and Jordan and Krishnamoorthy's interval.

- When the two variances are widely apart, they recommended the use of the inverse χ^2-method followed by Jordan and Krishnamoorthy (1996) and Fairweather (1972).

Examples

Example 1: Meier (1953) (reanalyzed in Jordan and Krishnamoorthy, 1996) considered four experiments about the percentage of albumin in plasma protein in human subjects

Percentage of albumin in plasma protein

Experiment	n_i	Mean	Variance
A	12	62.3	12.986
B	15	60.3	7.840
C	7	59.5	33.433
D	16	61.5	18.513

Examples

Interval estimates for μ in the albumin example

Method	95% CI on μ
Large Sample, $\text{CI}_1(\mu)$	60.99 ± 0.99
Sinha, $\text{CI}_2(\mu)$	60.99 ± 1.11
Meier, $\text{CI}_3(\mu)$	60.99 ± 1.13
Hartung, $\text{CI}_4(\mu)$	60.99 ± 1.66
Cohen & Sackrowitz (1984), $\text{CI}_5(\mu)$	60.82 ± 1.68
Cohen & Sackrowitz (1984), $\text{CI}_6(\mu)$	60.78 ± 1.58
Fairweather (1972), $\text{CI}_7(\mu)$	61.04 ± 1.15
Jordan & Krishnamoorthy (1996),$\text{CI}_8(\mu)$	61.00 ± 1.44
Inverse Normal, $\text{CI}_9(\mu)$	61.00 ± 1.31
Fisher, $\text{CI}_{10}(\mu)$	61.00 ± 1.42

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Selenium in nonfat milk powder

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2). Radiochemical	14	109.5	2.729
Isotope dilution mass spectrometry	8	113.25	33.640

Examples

Interval estimates for μ in the Selenium example

Method	95% CI on μ
Large Sample, $\text{CI}_1(\mu)$	109.6 ± 0.80
Sinha, $\text{CI}_2(\mu)$	109.6 ± 0.89
Meier, $\text{CI}_3(\mu)$	109.6 ± 0.90
Hartung, $\text{CI}_4(\mu)$	109.6 ± 1.71
Cohen & Sackrowitz (1984), $CI_5(\mu)$	109.5 ± 1.38
Cohen & Sackrowitz (1984), $CI_6(\mu)$	109.5 ± 1.27
Fairweather (1972), $CI_7(\mu)$	109.7 ± 1.11
Jordan & Krishnamoorthy (1996), $CI_8(\mu)$	109.6 ± 1.08
Inverse Normal, $CI_9(\mu)$	109.6 ± 0.93
Fisher, $CI_{10}(\mu)$	109.6 ± 1.09

R and SAS Programs

- SAS program: Common_mean.sas
  Graybill-Deal estimator and four approximate confidence intervals

- R program: Common_mean_approx.R
  Graybill-Deal estimator and four approximate confidence intervals

- R program: Common_mean_exact.R
  Exact confidence intervals

- Available from Guido Knapp's website
  `http://www.statistik.tu-dortmund.de/~knapp`

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