Information Theory in Volume Visualization

Ivan Viola
Today’s Menu

- Entropy
- Joint and Conditional Entropy
- Information Channel
- Mutual Information
- Entropy Rate
- Informational Divergence
Entropy

Random variable $X$ taking values in an alphabet $X$

$X: \{x_1, x_2, ..., x_n\}, p(x) = \Pr\{X = x\}, p(X) = \{p(x), x \in X\}$

Shannon entropy $H(X)$: uncertainty, information, homogeneity, uniformity

$$H(X) = - \sum_{x \in X} p(x) \log p(x) \equiv - \sum_{i=1}^{n} p(x_i) \log p(x_i)$$

Viewpoint entropy $H(v)$ based on triangle area ratio

$$H(v) = - \sum_{i=0}^{N_f} \frac{a_i}{a_t} \log \frac{a_i}{a_t},$$
View Selection for Set of Iso-Surfaces

- Decompose volume into feature sub-volumes
- Surface-based method for local optimal viewpoints
- Find global compromise between local optimal viewpoints

[Takahashi et al. 2005]
View Selection for Set of Iso-Surfaces

Calculate the weighted sum of the viewpoint entropies of the extracted iso-surfaces

Assign higher weights to feature iso-surfaces with opacity transfer functions

\[ E_{total} = w_1 E_1 + w_2 E_2 + w_3 E_3 + w_4 E_4 + w_5 E_5 \]

[Takahashi et al. 2005]

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View Selection for Set of Iso-Surfaces

[Takahashi et al. 2005]
Derived Entropy Measures

- **Discrete random variable** $Y$ in an alphabet $\mathcal{Y}$
  \[
  \mathcal{Y} = \{y_1, y_2, \ldots, y_n\}, \quad p(y) = \Pr\{Y = y\}
  \]

- **Joint entropy** $H(X,Y)$
  \[
  H(X,Y) = - \sum_{x \in X} \sum_{y \in Y} p(x,y) \log p(x,y)
  \]

- **Conditional entropy** $H(Y|X)$
  \[
  H(Y \mid X) = \sum_{x \in X} p(x) H(Y \mid x) = - \sum_{x \in X} p(x) \sum_{y \in Y} p(y \mid x) \log p(y \mid x)
  \]
  \[
  = - \sum_{x \in X} \sum_{y \in Y} p(x,y) \log p(y \mid x)
  \]
  \[
  p(x,y) = p(x)p(y \mid x) = p(y)p(x \mid y)
  \]
View Selection for Volume Data

- 3D or 4D scalar fields (over time)
- Probability function $q_j$
  
  $$q_j \equiv q_j(V) = \frac{1}{\sigma} \cdot \frac{v_j(V)}{W_j}$$
  
  where, $\sigma = \sum_{j=0}^{J-1} \frac{v_j(V)}{W_j}$

- Importance $W_j$: voxel opacity and information content
- View selection for time-series uses Conditional entropy

$$H(X) = H(X_1) + H(X_2 | X_1) + \ldots + H(X_n | X_{n-1}); H(X_2 | X_1) = - \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} p(x_1, x_2) \log p(x_2 | x_1)$$

[Bordoloi and Shen 2005]
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Information Channel

Communication or information channel $X \rightarrow Y$

$X \rightarrow Y$

$p(x_1) \quad p(y_1 | x_1) \quad p(y_2 | x_1) \quad \ldots \quad p(y_m | x_1)$

$p(x_2) \quad p(y_1 | x_2) \quad p(y_2 | x_2) \quad \ldots \quad p(y_m | x_2)$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$p(x_n) \quad p(y_1 | x_n) \quad p(y_2 | x_n) \quad \ldots \quad p(y_m | x_n)$

$p(y) = \sum_{x \in X} p(x)p(y | x)$

$p(Y | x) = \frac{p(y | x)}{p(X | x)}$

$p(Y) = \sum_{x \in X} p(X | x)p(Y | x)$
Mutual Information

Mutual information $I(X,Y)$: shared information, correlation, dependence, information transfer

$$I(X,Y) = H(Y) - H(Y | X) = \sum_{x \in X} \sum_{y \in Y} p(x, y) \log \frac{p(x, y)}{p(x)p(y)}$$

$$= \sum_{x \in X} p(x) \sum_{y \in Y} p(y | x) \log \frac{p(y | x)}{p(y)}$$

Viewpoint Mutual Information $I(v,O)$: dependence of viewpoint $v$ on set of objects $O$

$$I(V,O) = \sum_{x \in X} \sum_{y \in Y} p(v,o) \log \frac{p(v,o)}{p(v)p(o)} = \sum_{v \in V} p(v) \sum_{o \in O} p(o | v) \log \frac{p(o | v)}{p(o)} = \sum_{v \in V} p(v) I(v,O)$$
View Selection for Volumetric Objects

- Characteristic view
- Emphasis of focus object
- Guided navigation between characteristic views

[Viola et al. 2006]
Viewpoint Estimation

object-space distance weight

visibility estimation

image-space weight

information-theoretic framework for optimal viewpoint estimation

\[
I(v_i, O) = \sum_{j} p(o_j|v_i) \log \frac{p(o_j|v_i)}{p(o_j)}
\]

[Viola et al. 2006]
Focus of Attention

object selection by user

importance distribution

up-vector information

characteristic viewpoint

viewpoint transformation

cut-away and level of ghosting

focus discrimination

[Viola et al. 2006]
View Selection for Volumetric Objects

[Viola et al. 2006]
Iso-Surface Similarity Maps

- Compare iso-surfaces through evaluating mutual information of their distance volume
  - $X$ and $Y$ are **independent**: $I(X, Y) = 0$
  - $X$ and $Y$ are **identical**: $I(X, Y) = H(X) = H(Y)$
Iso-Surface Similarity Maps

Normalized measure

$$\hat{I}(X, Y) = \frac{2I(X, Y)}{H(X) + H(Y)}$$

[Bruckner and Möller 2010]
Iso-Surface Similarity Maps

Selection of characteristic iso-surfaces

[Bruckner and Möller 2010]
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Entropy Rate

- Shannon entropy

- Joint entropy of $L$ vector

- Entropy rate represents the average information content per symbol in a stochastic process

$$H(X) = - \sum_{x \in X} p(x) \log p(x)$$

$$H(X^L) = - \sum_{x^L \in X^L} p(x^L) \log p(x^L)$$

$$h = \lim_{L \to \infty} \frac{H(X^L)}{L} = \lim_{L \to \infty} (H(X^L) - H(X^{L-1}))$$
Similarity-Based Exploded Views

A two step process is proposed to automatically obtain the partitioning planes:

- Explosion axis: selection of the most structured view
- Partitioning of the data: slices are grouped according to the maximization of a similarity criterion

[Ruiz et al. 2008]
Structured View measured through Entropy Rate captures the randomness or unpredictability of a system
Bottom-up Grouping: group the most similar slices or slabs through normalized mutual information degree of similarity or shared information between two slices or slabs
Similarity-Based Exploded Views

[Ruiz et al. 2008]
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Informational Divergence

- **Informational divergence** (relative entropy or Kullback-Leibler distance)

\[ D_{KL}(p,q) : \text{how much } p \text{ is different from } q \text{ (on a common alphabet } X) \]

\[ D_{KL}(p,q) = \sum_{x \in X} p(x) \log \frac{p(x)}{q(x)} \]

- Convention: \( 0 \log 0/q = 0 \) and \( p \log p/0 = \infty \)

- \( D_{KL}(p,q) \geq 0 \)

- Not a true metric or distance

- Mutual information is a special case

\[ I(X,Y) = D_{KL}(p(X,Y), p(X)p(Y)) \]
Transfer Functions for Scalar Fields

Default Transfer Function

Visibility Distribution

New Transfer Function

Objective Function

Target Distribution

Optimizer

[Ruiz et al. 2011]
Transfer Functions for Scalar Fields

- **Target Function**: Intuitive specification of visual prominence for density values
- **Minimize informational divergence** \((F(A))\) between the average projected visibility distribution from viewpoints and a target distribution
- **Optimizer**: Steepest Gradient Descent

\[A^t = A^{t-1} - s^{t-1} \nabla F(A)\]

\[\nabla F(A) = \left( \frac{\partial F(A)}{\partial \alpha_0}, \frac{\partial F(A)}{\partial \alpha_1}, \ldots, \frac{\partial F(A)}{\partial \alpha_{n-1}} \right)\]
Transfer Functions for Scalar Fields

Ruiz et al. 2011

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Thank you!

More Information in recent book

Online:

folk.uib.no/ivi081/SA2011/notes.pdf